MOMC Regional Mathematical Olympiad Mock
Orange 2

Time: 3 Hours **October 24, 2023**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**
- **Problems collected by Agamjeet Singh**

1. Let $a_1, a_2, a_3, \ldots$ be the sequence of real numbers defined by $a_1 = 1$ and

$$a_m = \frac{1}{a_1^2 + a_2^2 + \ldots + a_{m-1}^2} \qquad \text{for } m \geq 2.$$

Determine whether there exists a positive integer N such that

$$a_1 + a_2 + \ldots + a_N > 2023^{2023}.$$

2. In $\triangle ABC$, $\angle BAC = 60^\circ$, point D lies on side BC, O_1 and O_2 are the centers of the circumcircles of $\triangle ABD$ and $\triangle ACD$, respectively. Lines BO_1 and CO_2 intersect at point P. If I is the incenter of $\triangle ABC$ and H is the orthocenter of $\triangle PBC$, then prove that the four points B, C, I, H are on the same circle.

3. Let Q be a set of permutations of $1, 2, ..., 100$ such that for all $1 \leq a, b \leq 100$, a can be found to the left of b and adjacent to b in at most one permutation in Q. Find the largest possible number of elements in Q.

4. The function $f : R \to R$ is defined as $f(x) = \lfloor x^2 \rfloor + \{x\}$. Call a real number *regional* if it doesn't lie in the range of f. Show that there exists an infinite arithmetic progression of distinct regional positive rational numbers, which all have denominator 3 when written in lowest form.

5. Find all integers n such that there exists a concave pentagon which can be dissected into n congruent triangles.

6. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sqrt[4]{\frac{\sqrt{3}}{a} + 6\sqrt{3}b} + \sqrt[4]{\frac{\sqrt{3}}{b} + 6\sqrt{3}c} + \sqrt[4]{\frac{\sqrt{3}}{c} + 6\sqrt{3}a} \leq \frac{1}{abc}$$

———0———