MOMC Regional Mathematical Olympiad Mock
Pasta 1

Time: 3 Hours **September 14, 2024**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let $ABCD$ be a cyclic quadrilateral. A circle passing through A and B meets AC and BD at points E and F respectively. The lines AF and BC meet at point P, and the lines BE and AD meet at point Q. Prove that PQ is parallel to CD.

2. Peter and Bob play a game on a $n \times n$ chessboard. At the beginning, all squares are white apart from one black corner square containing a rook. Players take turns to move the rook to a white square and recolour the square black. The player who can not move loses. Peter goes first. Who has a winning strategy?

3. Prove that for all natural numbers n,

$$\sum_{k=1}^{n^2} \left\{ \sqrt{k} \right\} \leq \frac{n^2-1}{2}.$$

Here, $\{x\}$ denotes the fractional part of x.

4. Determine all positive integers n for which there exist positive divisors a, b, c of n such that $a > b > c$ and $a^2 - b^2$, $b^2 - c^2$, $a^2 - c^2$ are also divisors of n.

5. Let H be the orthocenter of an acute-angled triangle ABC; E, F be points on AB, AC respectively, such that $AEHF$ is a parallelogram; X, Y be the common points of the line EF and the circumcircle ω of triangle ABC; Z be the point of ω opposite to A. Prove that H is the orthocenter of triangle XYZ.

6. An integer $m > 1$ is called *rich* if for any positive integer n, there exist positive integers x, y, z such that $n = mx^2 - y^2 - z^2$. An integer $m > 1$ is *poor* if it is not rich.

   - Show that there exist infinitely many poor integers.
   - Do there exist infinitely many rich integers?

———0———